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EXTENDING AN OPERATOR FROM A HILBERT SPACE TO A LARGER HILBERT SPACE, SO AS TO REDUCE ITS SPECTRUM

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ABSTRACT

In our earlier paper [1] we showed that given any element x of a commutative unital Banach algebra A, there is an extension A' of A such that the spectrum of x in A' is precisely the essential spectrum of x in A. In [2], we showed further that if T is a continuous linear operator on a Banach space X, then there is an extension Y of X such that T extends continuously to an operator $T^{\sim}$ on Y, and the spectrum of $T^{\sim}$ is precisely the approximate point spectrum of T. In this paper we take the second of these results, and show further that if X is a Hilbert space then we can ensure that Y is also a Hilbert space; so any operator T on a Hilbert space X is the restriction to one copy of X of an operator $T^{\sim}$ on $X \oplus X$, whose spectrum is precisely the approximate point spectrum of T. This result is "best possible" in the sense that if $\hat{T}$ is any extension to a larger Banach space of an operator T, it is a standard exercise that the approximate point spectrum of T is contained in the spectrum of $\hat{T}$.

§1. Introduction

The classical example of inverse producing extensions is surely the right shift operator, in the following way. Let X be a Hilbert space with orthonormal basis $\{e_i : i \geqq 0\}$, and let T be the right shift operator $T: e_i \rightarrow e_{i+1}$. It is well known that T has norm 1, the spectrum of T is the unit disk $\{z \in \mathbf{C}: |z| \leqq 1\}$, but the approximate point spectrum of T is just the unit circle $\{z \in \mathbf{C}: |z| = 1\}$. We can eliminate the open unit disk from the spectrum of T; to do this, it suffices to add on new orthonormal basis vectors $\{e_i : i \in \mathbf{Z},\ i < 0\}$, and extend T as

$$\hat{T}: e_i \rightarrow e_{i+1} \qquad (\text{all } i \in \mathbf{Z}).$$

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$\hat{T}$ has an obvious inverse $\hat{T}^{-1}$: $e_i \to e_{i-1}$, and indeed, the whole of the open unit disk is removed from the spectrum of $\hat{T}$. The object of this paper is to mimic this result as closely as possible in the case of a general continuous operator on X.

In order to do this we are going to draw on the results of [2], in which we established that there must be some extension of X (not necessarily another Hilbert space) to which T extends and loses all but its approximate point spectrum. Let us begin with some definitions.

DEFINITION 1.1. Let Ω denote the collection of all countable ordinals.

DEFINITION 1.2. Given the operator T on a Hilbert space X, let $\sigma_{\mathrm{ap}}(T)$ be the approximate point spectrum of T;

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbf{C}: \inf\{\|(\lambda I - T)x\|: x \in X, \|x\| = 1\} = 0\}.$$

(Note that in [2], §2.2 we call this set the "essential spectrum" of T.)

DEFINITION 1.3. Let us choose, once and for all, a collection of bounded open neighbourhoods $(U_i)_{i=1}^{\infty}$ of $\sigma_{\mathrm{ap}}(T)$ with the following properties.

(1) For each $i \in \mathbf{N}$, $U_i = \bar{U}_{i+1}$ (where $\bar{U}$ denotes the closure of U).

(2) Every component of U intersects $\sigma_{\mathrm{ap}}(T)$.

(3) $\bigcap_{i=1}^{\infty} U_i = \sigma_{\mathrm{ap}}(T)$.

Analogous definitions are in Sections 2.1 and 2.2 of [2].

At the corresponding point in [2] we proceeded to embark on the definitions of extensions X_α of X, one for each ordinal $\alpha \in \Omega$. (Briefly, each $X_{\alpha+1} = l_\infty(X_\alpha)/c_0(X_\alpha)$ and when α is a limit ordinal, X_α is the completion of the direct limit of the ones preceding it, with linking maps defined in an obvious way.) It is not necessary for our purposes to use these larger spaces; we merely quote the following useful definitions and results for the ordinal $\alpha = 1$.

DEFINITION 1.4. With the notation of [2], we have $X_1 = X$ ([2], Definition 2.8(1)), and for each $n \in \mathbf{N}$ we define $V_1^{(n)} = V^{(n)}$ to be the vector space of all bounded analytic functions $U_n \to X$ ([2], Definition 3.1). On $V^{(n)}$ we impose the supremum norm $\|\cdot\|_1^{(n)} = \|\cdot\|^{(n)}$,

$$\|f\|^{(n)} = \sup_{z \in U_n} \|f(z)\|_X.$$

Given a sequence $\varepsilon = (\varepsilon_i)_{i=1}^{n}$ we may define another norm $\|\cdot\|^{(\varepsilon)}$ on $V_1^{(n)}$ as follows ([2], Definition 3.2):

$$\|f\|^{(\varepsilon)}=\inf\left\{\|c\|_X+\sum_{i=1}^{n}\|f_i\|^{(i)}: c\in X, f_i\in V^{(i)},\right.$$
$$\left. c+\sum_{i=1}^{n}\varepsilon_i f_i(z)=f(z) \text{ for all } z\in U_n\right\}.$$

We observe that, given $\varepsilon_n>0$, the two norms are certainly equivalent.

We are interested (as in [2], Definition 3.3) in certain quotient spaces $Y^{(n)}=V^{(n)}/Z^{(n)}$, where $Z^{(n)}$ is the norm closed subspace of $V^{(n)}$ generated by those analytic functions f which satisfy

$$f(z)=zg(z)-T\circ g(z)=(zI-T)\circ g(z)$$

for all $z\in U_n$, where g is another element of $V^{(n)}$.

Then we define (as in [2], §3.3(2)) the map

$$\Psi^{(n)}: X\to(Y^{(n)},\|\cdot\|^{(\varepsilon)})$$

which sends $x\in X$ into $Y^{(n)}$ as the equivalence class of the constant function $x\in V^{(n)}$. The sequence $(\varepsilon_i)_{i=1}^{n}$ is said to be admissible (for the ordinal 1) if $\Psi^{(n)}: X\to(V^{(n)},\|\cdot\|^{(\varepsilon)})/Z^{(n)}$ is an isometry. It is a consequence of [2], Theorem 3.5, that there is a decreasing sequence of strictly positive constants $(\varepsilon_i)_{i=1}^{\infty}$ such that for all n the sequence $(\varepsilon_i)_{i=1}^{n}$ is admissible for the ordinal 1.

Let us choose such a sequence $(\varepsilon_i)_{i=1}^{\infty}$, and proceed to define some Euclidean extensions of X.

When dealing with the Hilbert space there is a slight technical blemish in the final result; if X is to be isometrically embedded in Y we cannot ensure that T extends to Y with exactly the same norm. Instead, the best we can do is to choose a fixed $\eta\in(0,1)$ and ensure that X is embedded isometrically in the Hilbert space Y, and T extends to an operator $T^{\sim}$ on Y, with

$$\|T^{\sim}\|\leqq(1-\eta)^{-1}\cdot\|T\|.$$

Suppose, then, that $\eta>0$ is given. Instead of taking the supremum norm $\|\cdot\|^{(i)}$ on $V^{(i)}$, let us consider the Euclidean norm

$$\|f\|_{i,2}=\left(\frac{1}{\lambda(U_i)}\cdot\int_{U_i}\|f(z)\|^2\,d\lambda(z)\right)^{1/2},$$

where λ is Lebesgue measure on $\mathbf{C}$. If X is a Hilbert space then $\|\cdot\|_{i,2}$ is a Euclidean norm on $V^{(i)}$.

Let us write $\Delta_n=d(U_{n+1},\partial U_n)$. Then for each $z\in U_{n+1}$, $f\in V^{(n)}$ and $r\in(0,\Delta_n)$ we have, by the Cauchy integral formula,

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta$$

hence

$$f(z)=\frac{1}{\pi\Delta_n^2}\cdot\int_{r=0}^{\Delta_n}\int_{\theta=0}^{2\pi} f(z+re^{i\theta})\cdot rdrd\theta$$
$$=\frac{1}{\pi\Delta_n^2}\cdot\int_{w\in B(z,\Delta_n)} f(w)d\lambda(w);$$

$$\|f\|^{(n+1)}=\sup_{z\in U_{n+1}}\|f(z)\|$$
$$=\sup_{z\in U_{n+1}}\frac{1}{\pi\Delta_n^2}\cdot\left\|\int_{w\in B(z,\Delta_n)} f(w)d\lambda(\omega)\right\|$$
$$\leqq\frac{1}{\pi\Delta_n^2}\cdot\int_{w\in U_n}\|f(w)\|\,d\lambda(w)$$
$$\leqq\frac{(\lambda(U_n))^{1/2}}{\pi\Delta_n^2}\cdot\left(\int_{w\in U_n}\|f(w)\|^2 d\lambda(w)\right)^{1/2}$$

(by the Jordan–Hölder inequality)

$$=\frac{\lambda(U_n)}{\pi\Delta_n^2}\cdot\|f\|_{n,2}.$$

So the supremum norm $\|f\|^{(n+1)}$ satisfies

$$\|f\|^{(n+1)}\leqq\alpha_n\cdot\|f\|_{n,2} \tag{6.1}$$

where α_n is a fixed constant.

Now if $(\varepsilon_i)_{i=1}^{\infty}$ is an admissible sequence, we know that if we define our function space $V=\bigcup_{i=1}^{\infty}V^{(i)}$ as in the proof of [2], Theorem 5, we have, for all $c\in X$,

$$\|c\|_X=\|c\|_V$$
$$=\inf\Big\{\|d\|_X+\sum_{i=1}^{n}\|f_i\|^{(i)}:\ n\in\mathbf{N},\ f_i\in V^{(i)},\ d\in X$$
$$\text{and for some } g\in V^{(n)} \text{ we have, for all } \lambda\in U_n,$$
$$d+\sum_{i=1}^{n}\varepsilon_i f_i(\lambda)+(\lambda I-T)\circ g(\lambda)=c\Big\}$$
$$\leqq\inf\Big\{\|d\|_X+\sum_{i=2}^{n}\|f_i\|^{(i)}:\ d+\sum_{2}^{n}\varepsilon_i f_i+(\lambda I-T)\circ g\equiv c\ \text{on}\ U_n\Big\}$$

$$(6.2)\qquad \leqq \inf\Big\{\|d\|_X+\sum_{i=2}^{n}\alpha_{i-1}\|f_i\|_{i-1,2}:\ d\in X,\ n\in\mathbf{N},$$

$$\text{each } f_i\in V^{(i-1)}, \text{ and } d+\sum_{2}^{n}\varepsilon_i f_i+(\lambda I-T)\circ g\equiv c \text{ on } U_n\Big\} \text{(by (6.1))}$$

$$\leqq\Big(\sum_{0}^{n-1}\eta^{j}\Big)^{1/2}\cdot\inf\Big\{\Big(\|d\|_X^2+\sum_{i=2}^{n}\eta^{1-i}\alpha_{i-1}^2\|f\|_{i-1,2}^2\Big)^{1/2}:$$

$$d+\sum_{2}^{n}\varepsilon_i f_i+(\lambda I-T)\circ g\equiv c \text{ on } U_n\Big\}$$

(by the Jordan–Hölder inequality)

$$\leqq(1-\eta)^{-1}\cdot\|c\|_{V,2}$$

where for all $f\in V$, we define

$$(6.3)\qquad \begin{aligned}\|f\|_{V,2}=\inf\Big\{&\Big(\|d\|_X^2+\sum_{i=2}^{n}\eta^{1-i}\alpha_{i-1}^2\|f_i\|_{i-1,2}^2\Big)^{1/2}:\\ &d\in X,\ f_i\in V^{(i-1)}\ (i=2,\dots,n),\text{ and for some}\\ &g\in V^{(n)},\text{ we have for all }\lambda\in U_n,\\ &d+\sum_{i=2}^{n}\varepsilon_i f_i(\lambda)+(\lambda I-T)\circ g(\lambda)=f(\lambda)\Big\}.\end{aligned}$$

We claim that the expression $\|\cdot\|_{V,2}$ is a Euclidean seminorm on V. Let $V^{\sim}$ be the quotient space V/Z where $Z=\{f\in V:\|f\|_{V,2}=0\}$. Consider the cartesian product $X\bigoplus_{i=1}^{\infty}V^{(i)}$. Those elements of this cartesian product which have finite norm

$$\|(d,f_2,f_3,f_4,\dots)\|=\Big(\|d\|_X^2+\sum_{2}^{\infty}\eta^{1-i}\alpha_{i-1}^2\|f_i\|_{i-1,2}^2\Big)^{1/2}$$

are a normed vector space with a Euclidean norm. $V^{\sim}$ is a quotient space of this normed vector space in an obvious way, so it too is a Euclidean space. Let $\hat{V}$ be the completion of $V^{\sim}$. $\hat{V}$ is a Hilbert space, and we claim that X is isomorphically embedded in $\hat{V}$ as the equivalence classes of the constant functions of V.

For if $f\in V$ is a constant c, in view of (6.2) we have

$$\|c\|_{V,2}\geqq(1-\eta)\|c\|_X.$$

However, it is obvious from the definition (6.3) of $\|\cdot\|_{V,2}$ that

$$\|c\|_{V,2}\leqq\|c\|_X.$$

So X is embedded up to $(1-\eta)^{-1}$ isomorphism in $(\hat{V},\|\cdot\|_{V,2})$. Moreover T extends to $\hat{V}$ as the operator

$$T^{\sim} : \hat{V} \to \hat{V}$$

$$: [f] \to [T \circ f]$$

which, given the definition (6.3) of the norm on V, clearly has norm less than or equal to $\| T \|$.

All but the approximate point spectrum of T is eliminated, because, as in [2], proof of Theorem 5, if $\mu \notin \sigma_{ap}(T)$ then the operator

$$R_\mu : [f] \to [g] \qquad \text{where } g(\lambda) = (\mu - \lambda)^{-1} f(\lambda)$$

gives a continuous inverse for $\mu I - T^{\sim}$. We omit the proof of this, since the details are very similar to those in [2], §5. So $\hat{V}$ is a Hilbert space in which X is embedded up to $(1-\eta)^{-1}$ equivalence; $T^{\sim}$ is a continuous extension of T to $\hat{V}$, and the spectrum of $T^{\sim}$ is precisely the approximate point spectrum of T. It is trivial to adjust the norm on $\hat{V}$ so that X is embedded isometrically (let W be the orthogonal complement of X in $\hat{V}$ and consider the Euclidean norm

$$\| f \| = (\| w \|_{V,2}^2 + \| x \|_X^2)^{1/2}$$

where $w \in W$, $x \in X$, $w + x = f$), but then $\| T^{\sim} \|$ may be increased by a factor of $(1-\eta)^{-1}$. So we have the following theorem.

THEOREM. *Given a Hilbert space* X and $T \in L(X)$, $\varepsilon \in (0,1)$, *there is a Hilbert space* $Y \supset X$ *and an operator* $T^{\sim} \in L(Y)$ *which extends* T, *such that* $\| T^{\sim} \| \leqq \| T \| (1 + \varepsilon)$, *and the spectrum of* $T^{\sim}$ *is the approximate point spectrum of* T.

It is trivial that the (infinite) dimension of Y/X is the same as that of X, so the theorem may alternatively be written as follows:

THEOREM. *If* T *is a continuous linear operator on a Hilbert space* H, *and if* $\eta \in (0,1)$, *then* T *is the restriction to one copy of* H *of an operator* $T^{\sim}$ *on the Hilbert space* $H \oplus H$, *such that*

$$\| T^{\sim} \| \leqq (1-\eta)^{-1} \cdot \| T \|,$$

and the spectrum of $T^{\sim}$ *is precisely the approximate point spectrum of* T.

REFERENCES

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